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**Abstract.** An algebraic approach to the solution of the Klein–Gordon equation is described for the case of a charged particle in the presence of plane-wave electromagnetic radiation. From an examination of the commutation relations between $P_\mu = -i(\partial/\partial x_\mu)$ and $A_\nu$, $P \cdot A$, $A \cdot A$, etc, one finds a new set of 'translation' operators $\Pi_\mu$ which commute with the total 'Hamiltonian'. We then construct a representation of the Poincaré group out of the $\Pi_\mu$ and their canonically conjugate 'coordinates' $Q_\nu$. The solutions are shown to correspond to the spin zero, mass $m$ representation of the restricted Poincaré group. Applications of the technique to other quantum-mechanical problems are also briefly discussed.

## 1. Algebraic formulation of the problem

The object of this paper is to describe an algebraic approach to the solution of relativistic quantum-mechanical problems involving the interaction of a charged particle with various types of external electromagnetic fields. In order to illustrate the method, we use it to solve the Klein–Gordon equation for a charged particle interacting with a classical (unquantized) plane-wave radiation field—a problem which is sufficiently nontrivial to illustrate the power of a purely algebraic approach, even though the solution of this particular problem is well known (see, for example, Volkov 1935, Dirac 1946, Taub 1949, Nikishov and Ritus 1964, Brown and Kibble 1964).

In this connection it should be noted that Chakrabarti (1968) was apparently the first to notice that the Volkov solutions imply the existence of a Poincaré algebra (the same algebra which we shall use to solve the problem). We also call the reader's attention to the relatively complete group-theoretical analysis of elementary particles in certain types of external electromagnetic fields given by Bacry *et al* (1970a, b) (see also Richard 1972). The work described below was completed before we learned of the earlier work by Chakrabarti and Bacry *et al*, and our work is certainly very similar in spirit to theirs. There is, however, one major difference between our algebraic method and their earlier work; we derive the solution from first principles by using algebraic methods only. Chakrabarti and other workers noted that, given the explicit form of the solutions, certain transformations must exist between the 'free' generators and the generators in the presence of the laser field. It is our hope that the algebraic method can be applied to related problems which have not yet been solved, and by using algebraic and/or group-theoretic techniques it will be possible to obtain exact analytic solutions. We therefore briefly discuss other applications of our method in § 2.

Our procedure is based on an investigation of the algebra satisfied by the generators of space–time translations, where the usual generators are regarded as being 'deformed' by the presence of the electromagnetic field, the 'deformation' being specified by the principle of minimal electromagnetic interaction (Gell-Mann 1956, especially § 3.3 on p 853). From this algebra a set of operators is identified which commute with the 'Hamiltonian' of the problem and obey the algebra of the generators of the Poincaré group. This being the case, we may classify the eigenstates of the 'Hamiltonian' according to their transformation properties under the Poincaré group. That is, the Hilbert space of eigenstates of the 'Hamiltonian' must be the carrier space for a unitary ray representation of the restricted Poincaré group $P_+^\uparrow$, which is equivalent to asking for true representations of the covering group of $P_+^\uparrow$ (Bargmann 1954). Since the irreducible unitary representations of $P_+^\uparrow$ have been classified by Wigner (1939) and Shirokov (1958), and we have an explicit realization of the generators in hand, we are able to explicitly determine which representations occur†.

The essence of the proposed algebraic method is best illustrated by first considering the case of the free Klein–Gordon equation‡.

For a free Klein–Gordon particle the Poincaré generators are the usual space–time translation operators $P_\mu = -\mathrm{i}(\partial/\partial x_\mu)$ and the six Lorentz operators $M^0_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu$. The invariant 'Hamiltonian' (wave operator) is given by§ $H_0 = P_\mu P_\mu$, and the eigenvalue problem is to determine the solutions of the wave equation $H_0\psi = -m^2\psi$. One easily verifies that $[P_\mu, H_0] = [M^0_{\mu\nu}, H_0] = 0$. The representation is first specified by the value of the mass-squared operator, which is, of course, $-m^2$. For fixed mass there are two representations, one each for the positive and the negative energy states. For these time-like (particle-like) representations a second Casimir operator is provided by $S^0_\mu S^0_\mu$ (for further details, see Shirokov 1958), where $S^0_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}M^0_{\nu\lambda}P_\sigma$ is the Pauli–Lubanski vector ($\epsilon_{\mu\nu\lambda\sigma}$ is the completely antisymmetric Levi–Civita tensor density). By symmetry $S^0_\mu$ vanishes identically. These two Casimir operators completely specify the representation as the mass $m$, spin zero representation of $P_+^\uparrow$.

Once the representation has been specified, the manifold of states is constructed by the little-group method of Wigner (1939) (this method has become known as the method of induced representations; see, for example, Mackey 1968). For positive mass representations, the little group is SU(2). In the rest frame of the particle a set of basis vectors is chosen which form an irreducible representation of the little group. States in an arbitrary frame are obtained by 'boosting' out of this special frame. The result for particle-like representations is that the state vectors are labelled by $p_\mu$, $J$, and $\lambda$, where $p_\mu p_\mu = -m^2$. Here $p_\mu$ denotes the eigenvalue of $P_\mu$, $J$ is the spin angular momentum of the representation of SU(2) which occurs (this is the label provided by $S^0_\mu S^0_\mu$), and $\lambda$ is related to the $z$ component of the spin (the exact relation depends on whether one works in the spin basis or the helicity basis).

† Wigner (1939) has shown that it is always possible to do this. In general, however, the reduction is a nontrivial problem.

‡ Before proceeding, however, it is important for the reader to recall the distinction between a representation and a realization (see, eg Miller 1968 or Haig *et al* 1963). The nontrivial unitary representations of $P_+^\uparrow$ (which are all infinite dimensional) are purely algebraic constructions which may be realized in any of the many isomorphic Hilbert spaces available. What we illustrate below is the well known fact that the only information contained in the free Klein–Gordon equation—over and above Poincaré invariance—is the specification that the representation be realized in the space of functions on space–time. We belabour this rather fine point somewhat, because we shall present two different realizations of the same representation of $P_+^\uparrow$ during the course of this article.

§ We use the nomenclature 'Hamiltonian' for this operator because this is the name reserved for the corresponding classical object in manifestly covariant canonical mechanics. This operator is to be distinguished from the energy operator $P_0$ which is normally called the Hamiltonian of the system. It should also be mentioned that the constructions given in this article (see also Beers and Nickle 1972a) will immediately provide the solution to the corresponding classical problem if the commutators are replaced by Poisson brackets.

For the case at hand, this procedure is trivial—since we have the identity representation of the little group ($J = 0$), and the states are uniquely labelled as simultaneous eigenstates of $P_\mu$. For the present realization of the $P_\mu$, the solutions are, of course, given by $\psi_{KG} \sim \exp(ip_\mu x_\mu)$. We have been unusually explicit in our discussion of this seemingly trivial problem for two reasons. First, for the specific problem used to illustrate the algebraic method (namely, the problem of a Klein–Gordon particle in the presence of plane-wave electromagnetic radiation), a formally identical procedure is used to construct the exact solutions. Secondly, for the more complicated problems which we hope to be able to solve by using this algebraic approach, more complicated representations of $P_+^\uparrow$ will occur (in particular, see the algebraic solution of the Dirac equation as given by Beers and Nickle 1972b, c).

Now let us consider the Klein–Gordon particle in the presence of a plane-wave electromagnetic field. In analogy to the free-particle case, the invariant 'Hamiltonian' is now given by

$$H = (P_\mu - qA_\mu)^2, \tag{1}$$

where the vector potential is given by

$$A_\mu(r, t) = a_\mu f(\phi) + b_\mu g(\phi), \tag{2}$$

where $f$ and $g$ are arbitrary functions of the argument $\phi \equiv k\,.\,x = \boldsymbol{k}\,.\,\boldsymbol{r} - \omega t$ and $k\,.\,a = k\,.\,b = a\,.\,b = 0$. The possibility of an algebraic solution then follows from a consideration of the so called '$P$ algebra', namely the complete set of commutation relations between $P_\mu$, $A_\nu$, $P\,.\,A$, $A\,.\,A$, and $H$ (for further details, see Beers and Nickle 1972a). In deriving this algebra it is essential to note that $k\,.\,P$ is a constant as far as the $P$ algebra is concerned, that is, $k\,.\,P$ commutes with $A_\mu$, $P\,.\,A$, and $A\,.\,A$ (due to the fact that $k\,.\,k = 0$ for the electromagnetic wave). We may therefore replace $k\,.\,P$ by the constant $\zeta = k\,.\,P \neq 0$ (this is also the essential reason why we are able to construct the new set of 'translation' operators from the elements of the $P$ algebra).

It is clear that the principle of minimal electromagnetic interaction has 'deformed' the usual Poincaré generators in such a way that they no longer generate the Poincaré group, that is, $[P_\mu - qA_\mu, P_\nu - qA_\nu] \neq 0$, etc. The algebraic approach involves using the '$P$ algebra' to construct a new set of 'translation' operators $\Pi_\mu$ which commute with each other and also commute with the invariant 'Hamiltonian' (1). By using the '$P$ algebra' together with the commutators of $x_\nu$ with those elements of the '$P$ algebra' which appear in the $\Pi_\mu$, we also find a new set of 'coordinates' $Q_\nu$ which are canonically conjugate to the $\Pi_\mu$, that is, $[\Pi_\mu, Q_\nu] = -i\delta_{\mu\nu}$. We then construct new Lorentz operators according to the rule: $M_{\mu\nu} = Q_\mu\Pi_\nu - Q_\nu\Pi_\mu$. The $\Pi_\mu$ and the $M_{\mu\nu}$ are then obviously the ten generators of a realization of the Poincaré group.

For the sake of brevity, we cite here only the explicit results for $\Pi_\mu$ and $Q_\nu$:

$$\Pi_\mu = P_\mu - (q/\zeta)k_\mu(P\,.\,A) + (q^2/2\zeta)k_\mu(A\,.\,A) \tag{3}$$

$$Q_\nu = x_\nu + (q/\zeta)\int A_\nu\,d\phi - (q/\zeta^2)k_\nu\int (P\,.\,A)\,d\phi + \tfrac{1}{2}(q/\zeta)^2 k_\nu \int (A\,.\,A)\,d\phi. \tag{4}$$

Here it should be noted that not only do the new translation generators commute with $H$, but in fact the Hamiltonian (1) now takes the form $H = \Pi_\mu\Pi_\mu$. One can also easily verify that the $Q_\mu$ commute with themselves; furthermore $[Q_\mu, H] = -2i\Pi_\mu$, and consequently the Lorentz operators $M_{\mu\nu}$ commute with $H$.

Using the 'new' set of group operators, we construct the Pauli–Lubanski 'spin' operator $S_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}M_{\nu\lambda}\Pi_\sigma$ and easily verify that it vanishes identically by symmetry. Thus we have $\Pi \,.\, \Pi = -m^2$ and $S \,.\, S = 0$. These two Casimir operators completely specify the solutions as the spin zero, mass $m$ representation of the restricted Poincaré group. The explicit position-space form of the solution follows easily from the fact that the states are uniquely labelled as simultaneous eigenstates of the operator (3), which gives the well known result

$$\psi_\kappa(x) \sim \exp\left\{\mathrm{i}\left(\kappa \,.\, x + (q/\zeta_\kappa)\kappa \,.\, \int A\,\mathrm{d}\phi - (q^2/2\zeta_\kappa)\int A \,.\, A\,\mathrm{d}\phi\right)\right\}$$

where $\kappa \,.\, \kappa = -m^2$ and $\zeta_\kappa$ denotes the eigenvalue of the operator $(k \,.\, P)$ when acting on the eigenfunction $\psi_\kappa(x)$ (here it is important to realize that the eigenvalue $\zeta_\kappa$ depends on the value of the four-vector $\kappa$ characterizing the exact solution $\psi_\kappa(x)$).

It should also be noted that the 'new' translation operators $\Pi_\mu$ play the same role as the integration constants (associated with the momentum) in the Hamilton–Jacobi solution of the classical problem, that is, they correspond to the constant Hamilton–Jacobi momenta of the classical solution (see, for example, Landau and Lifshitz 1962; the relation between the classical and quantum-mechanical treatments is discussed in detail in Appendix C of Brown and Kibble 1964). So in a certain sense the present approach can also be regarded as a purely algebraic method of solving the Hamilton–Jacobi equation†.

## 2. Other applications of the algebraic method

Here we wish to briefly indicate several problems where the algebraic method appears to be helpful.

### *2.1. Motion of a charged particle in the presence of a plane electromagnetic wave plus a static magnetic field whose direction is parallel to the direction of propagation of the wave*

The classical, relativistic equations of motion for this problem were first solved by Roberts and Buchsbaum (1964), and the Klein–Gordon and Dirac equations were subsequently solved by Redmond (1965). Although the solution of the Klein–Gordon equation has been investigated in detail by Redmond, the present approach leads to a very simple and elegant solution. In order to sketch the method, let us take

$$A = \hat{x}A(kz-\omega t) \tag{5}$$

$$\mathscr{A}_B = Bx\hat{y}. \tag{6}$$

Then the invariant Hamiltonian is given by

$$H = (\boldsymbol{P} - q\boldsymbol{A} - q\mathscr{A}_B)^2 + P_4^2 = (P_1 - qA)^2 + (P_2 - qBx)^2 + P_3^2 + P_4^2. \tag{7}$$

† One of us (BLB) has been able to prove that if $S$ is a solution of the Hamilton–Jacobi equation with potential $A_\mu$, then $\psi = \exp(S)$ is a solution of the Klein–Gordon equation in the gauge $A'_\mu = A_\mu + \partial_\mu\Lambda$, where $\Lambda$ is determined from the equation $(\partial_\mu - 2q\partial_\mu\Lambda)(\partial_\mu S - qA_\mu) + (\partial_\mu\Lambda - q\partial_\mu)\partial_\mu\Lambda = 0$. This suggests that it may be possible to establish a 'deeper' connection between the algebraic approach and the classical theory of contact transformations. This is one of our reasons for believing that the algebraic method (or appropriate 'variations on this theme') will be helpful in the solution of more complicated quantum-mechanical problems.

However, it is clear from the discussion given in § 1 that the Hamiltonian (7) can be rewritten in the form $H = \Pi_\mu \Pi_\mu - 2qB\Pi_2 x + (qBx)^2$, where the $\Pi_\mu$ are given by equation (3) with $k_\mu = (0, 0, k, i\omega)$ and $A_\mu = A(kz-\omega t)\delta_{\mu 1}$. We immediately see that the problem separates because the vector potential of the plane-wave field is a function of $kz-\omega t$ while the cross term involving $-2qB\Pi_2 x$ can be replaced by $-2qB\kappa_2 x$ (here $\kappa_2$ denotes the eigenvalue of the operator $\Pi_2$). Hence the problem reduces to

$$H = P_1^2 - 2qB\kappa_2 x + (qBx)^2 + \Pi_2^2 + \Pi_3^2 + \Pi_4^2,$$

which corresponds to a one dimensional harmonic oscillator plus motion of the type described in § 1.

This problem immediately suggests another simple problem which can be solved exactly. This is dealt with in the following section.

### 2.2. *Motion of a charged particle in the presence of a plane electromagnetic wave plus harmonic forces in the plane perpendicular to the direction of propagation of the wave*

Again assuming the vector potential of the plane-wave field to be given by equation (5), the invariant Hamiltonian is given by $H = (P_1 - qA)^2 + P_2^2 + P_3^2 + P_4^2 + \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2$. Once again we can use the results of § 1 to rewrite this Hamiltonian in the form: $H = \Pi_\mu \Pi_\mu + \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2$, that is, the problem 'separates' into a two dimensional harmonic oscillator plus a problem of the type solved in § 1.

These two examples suggest a more general type of problem which can easily be solved by using the algebraic approach.

### 2.3. *The general type of problem easily solved by the algebraic approach*

Suppose the Klein–Gordon equation can be solved in the presence of a given external field specified by the vector potential $A_\mu^{(2)}(x)$. Under what conditions can the same problem be solved in the presence of an additional plane-wave field specified by the vector potential $A_\mu^{(1)}(kx)$?

The invariant Hamiltonian is now given by

$$H = (P_\mu - qA_\mu^{(1)} - qA_\mu^{(2)})^2 = (P - qA^{(1)})^2 - 2q(P - qA^{(1)}) \, . \, A^{(2)} + q^2 A^{(2)} \, . \, A^{(2)}.$$

Now if $A^{(2)} \, . \, A^{(1)} = 0$ and $\Pi \, . \, A^{(2)} = P \, . \, A^{(2)}$ (where $\Pi_\mu$ is given by equation (3) with $A_\mu^{(1)}$ appearing on the right hand side), this Hamiltonian can be rewritten in the form

$$H = (\Pi - qA^{(2)})^2. \tag{8}$$

Furthermore, if $[\Pi_\mu, A_\nu^{(2)}] = [P_\mu, A_\nu^{(2)}]$ and if $A_\mu^{(2)}(x) = A_\mu^{(2)}(Q)$ (where $Q_\mu$ is given by equation (4) with $A_\mu^{(1)}$ appearing on the right hand side), then it is clear that if the solution to the problem defined by the Hamiltonian $(P - qA^{(2)})^2$ is known, then the solution of the problem defined by the Hamiltonian (8) can also be determined. So under the above restrictions, a solvable problem will remain solvable even in the presence of an additional plane-wave field.

### 2.4. *Motion of a charged particle in the presence of a plane electromagnetic wave plus a constant magnetic field pointing in any arbitrary direction*

As a fourth example of the utilization of this algebraic approach, we wish to describe a preliminary approach to the solution of the Klein–Gordon equation for a charged particle in the presence of both plane-wave electromagnetic radiation and a constant magnetic field pointing in any arbitrary direction.

It is convenient to write the invariant Hamiltonian in the form

$$H = (\boldsymbol{P} - qA - q\mathscr{A}_B)^2 + P_4^2, \tag{9}$$

where $A$ denotes the vector potential of the plane-wave field and the constant magnetic field is described by the vector potential $\mathscr{A}_B = -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B}^0$.

Choosing the $z$ axis to coincide with the direction of propagation of the plane wave, one can easily verify that the vector operator $\boldsymbol{K} = \boldsymbol{P} - P_0\hat{z} - \frac{1}{2}\boldsymbol{r} \times \boldsymbol{B}^0$ commutes with the Hamiltonian (9). Here it is essential to note that the components $K_i (i = 1, 2, 3)$ are not $c$ numbers, for example, $[K_1, K_2] = -iqB$—even though all three components commute with the Hamiltonian (9).

The algebraic method of attack is to consider the 'new' algebra generated by the $K_i$, the Hamiltonian (9), and the remaining elements of the so called '$P$ algebra' discussed in § 1 (that is, the elements of the 'new' algebra are the elements of the old '$P$ algebra', except that the new Hamiltonian (9) is used, plus the three new generators $K_i$). Hopefully this algebra can be identified without too much difficulty, and if its representations have already been investigated by mathematicians, then we will be able to solve the physical problem exactly. If the representations of the resulting algebra have not yet been investigated mathematically, then the problem will be to first classify the representations of the 'new' algebra. Although we have not yet completed our study of this particular problem, we wish to emphasize that in principle at least it appears to be solvable by using this type of approach.

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